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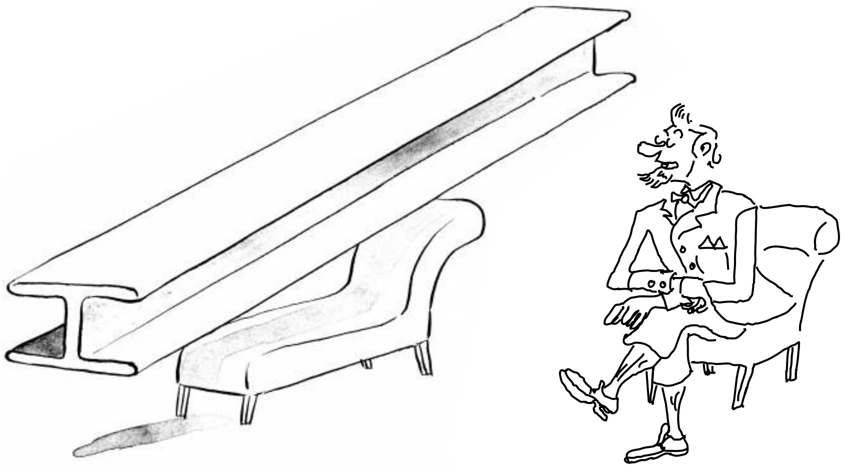
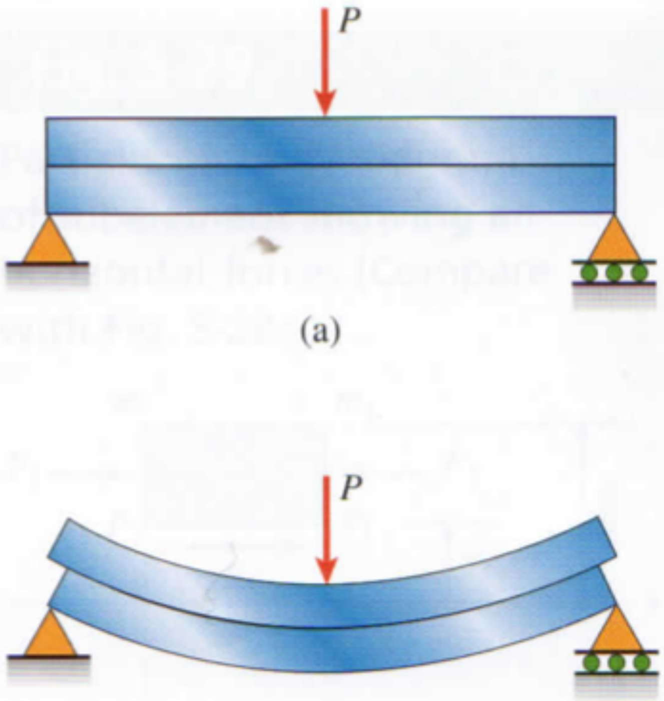


"You have clearly been under enormous stress."

Lecture 12:
Shear stresses in beams



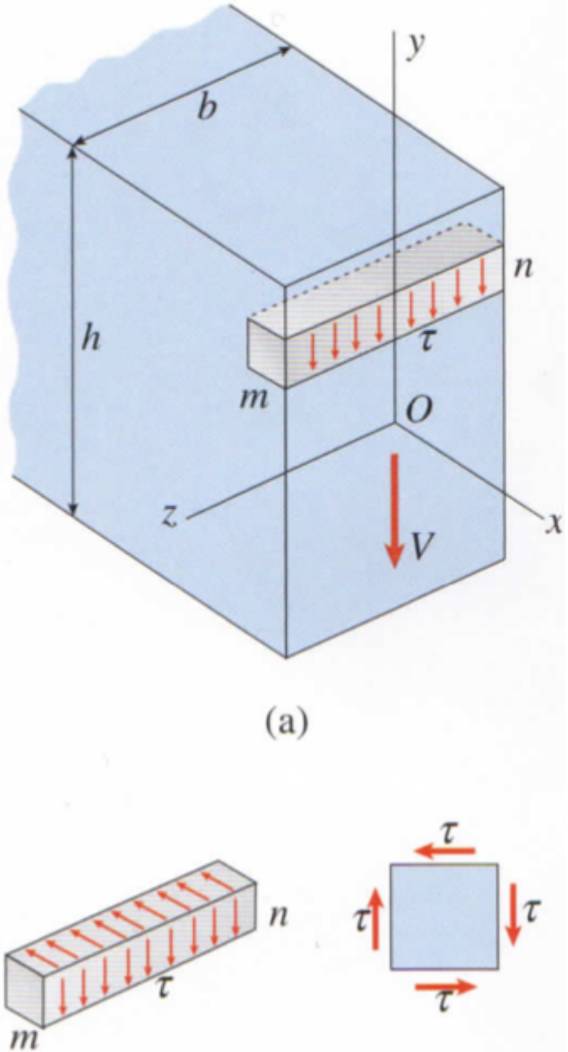
Shear stress in beams

We’ve seen that if a beam is in pure bending, the only stresses that act on the cross section are normal stresses

In non-uniform bending, we will have normal stresses and shear stresses

Assume two identical beams in bending. Under load, one beam will slide over the other. The force that is required to prevent this sliding is the longitudinal shear force.

Shear stress in beams



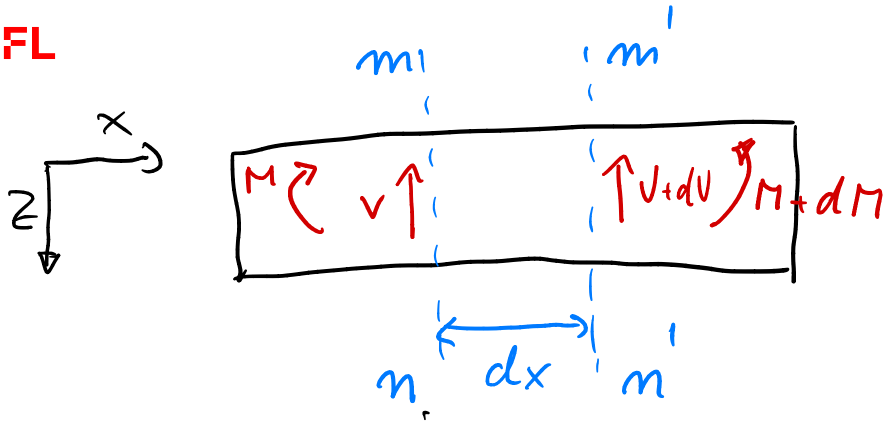
For the case of a rectangular cross-section we can assume:

- Shear stresses that act on a cross section are evenly distributed from one side to the other and act vertically
- Shear stresses acting on one side are accompanied by shear stresses of equal magnitude on a perpendicular face
- Therefore we will have vertical AND horizontal shear stresses

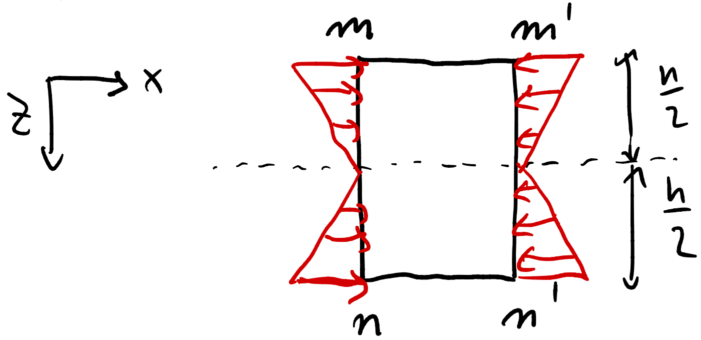
First result: at the top and at the bottom of the beam, the horizontal shear stresses are zero. Therefore also the vertical shear stresses are zero.

NON-UNIFORM BENDING

$\Rightarrow M, V$

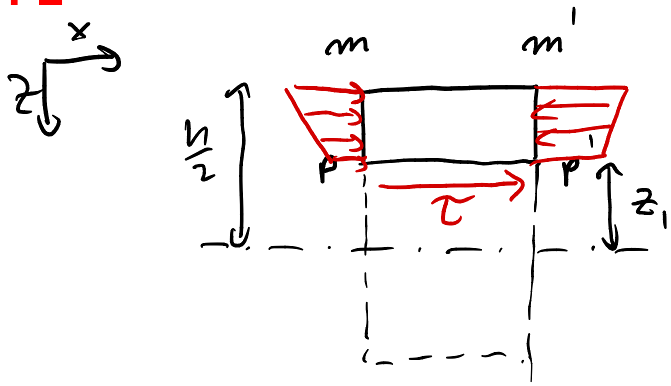


- START WITH NORMAL STRESSES



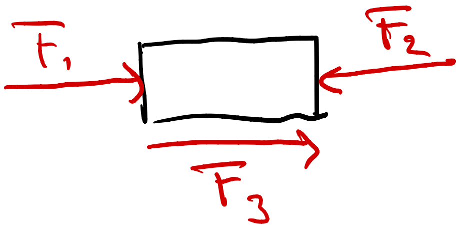
NORMAL STRESSES ON mn: $\sigma_1(z) = \dfrac{M \cdot z}{I}$

NORMAL STRESSES ON $m'n'$: $\sigma_2(z) = \dfrac{M + dM}{I} \cdot z$



IF THE MOMENT @ $mp \neq m'p' \Rightarrow \sum F_x \neq 0$

$\Rightarrow$ FOR $\sum F_x = 0 \Rightarrow$ THERE MUST BE A SHEAR FORCE



$$F_1 = \int_{z_1}^{h/2} \sigma_1 \, dA = \int_{z_1}^{h/2} \frac{M z}{I} \, dA$$

$$F_2 = \int_{z_1}^{h/2} \sigma_2 \, dA = \int_{z_1}^{h/2} \frac{(M + dM)}{I} z \, dA$$

$$F_1 + F_3 = F_2 \Rightarrow F_3 = F_2 - F_1$$

$$F_3 = \int \frac{M + dM}{I} z \, dA - \int \frac{M}{I} z \, dA = \int \frac{dM}{I} z \, dA = \frac{dM}{I} \int z \, dA$$

SINCE WE ASSUMED THAT τ IS EQUALLY DISTRIBUTED OVER WIDTH

$$F_3 = \tau \cdot A = \tau \cdot b \cdot dx \Rightarrow \tau = \frac{F_3}{b \cdot dx}$$

$$\tau = \underbrace{\frac{dM}{dx}}_{V} \cdot \frac{1}{Ib} \int z \, dA$$

$$\Rightarrow \tau = \frac{V}{I \cdot b} \underbrace{\int z \, dA}_{1^{ST} \text{ MOMENT OF AREA}}$$

$$\boxed{\tau = \frac{V}{Ib} \cdot Q}$$

SHEAR FORMULA

0-MOMENT OF AREA

$$\int_A 1 \cdot dA = A$$

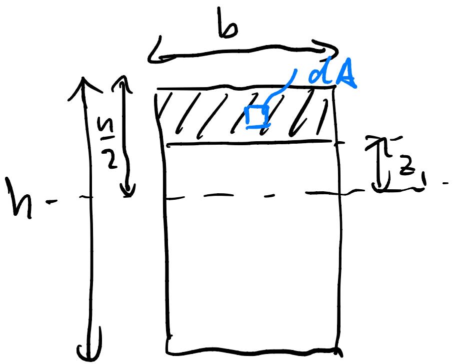
1^{ST} MOMENT OF AREA

$$\int_A z \, dA := Q$$

2^{ND} MOMENT OF AREA: $\int_A z^2 \, dA = I$

Georg Fantner

CALCULATE Q FOR A RECTANGULAR BEAM



$$Q = \int z\,dA = \int_{z_1}^{h/2} b \cdot z\,dz = \frac{b}{2}\left(\frac{h^2}{4} - z_1^2\right)$$

WITH SHEAR FORMULA WE GET:

$$\tau = \tau(z_1) = \frac{V}{2I}\left(\frac{h^2}{4} - z_1^2\right)$$

$\square$ @ $z_1 = \pm\frac{h}{2} \Rightarrow \tau = 0$

$\square$ @ $z_1 = 0 \Rightarrow \tau = \tau_{MAX} = \frac{V}{2I}\frac{h^2}{4} = \frac{Vh^2}{8I}$

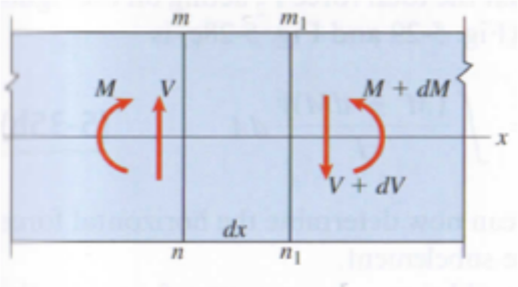
$I = \frac{bh^3}{12}$ FOR RECT. BEAM

$A = b \cdot h$

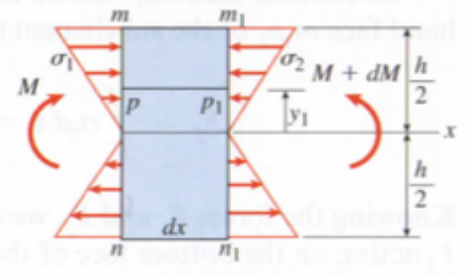
$$\boxed{\tau_{MAX} = \frac{3}{2}\frac{V}{A}}$$

$\Rightarrow$ τ_{MAX} is 50% LARGER THAN AVERAGE SHEAR STRESS.

Shear stress in beams



Side view of beam



Side view of element

- Assume a beam in non-uniform bending and take two vertical cross-sections a distance *dx* apart.
- Since we have non-uniform bending, we have both normal stresses and shear stresses.
- We want to find an expression that relates external forces to shear stresses
- Our approach:
  - Use the fact that shear stresses act both normal to the beam axis as well as parallel to the beam axis
  - Use the expression for normal stresses (flexure formula) to calculate force-equilibrium in the longitudinal direction in the presence of a moment gradient.

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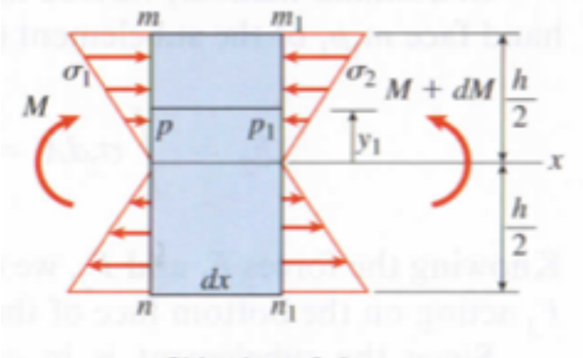
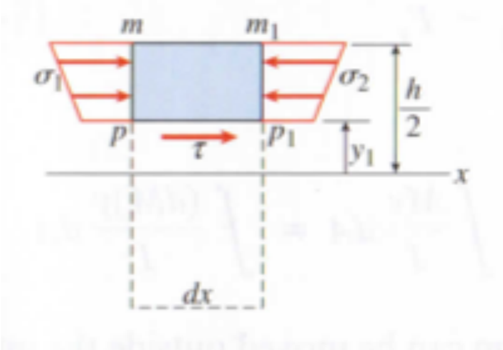
Shear stress in beams



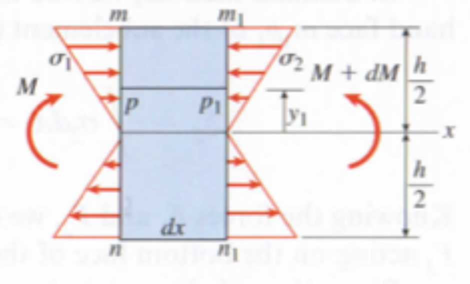
Side view of element



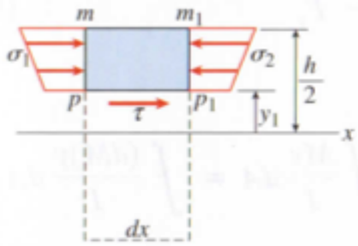
Side view of subelement

- We calculate the normal stresses on each cross-section of the element using the flexure formula
- The stresses vary linearly with y, being zero at y=0
- We know that shear stresses are stresses that occur between two sides of a cross-section. We make a horizontal cut through the element to look at the sub-element mm_1pp_1.
- Calculate the forces acting on the sub-element in the x-direction.

Shear stress in beams



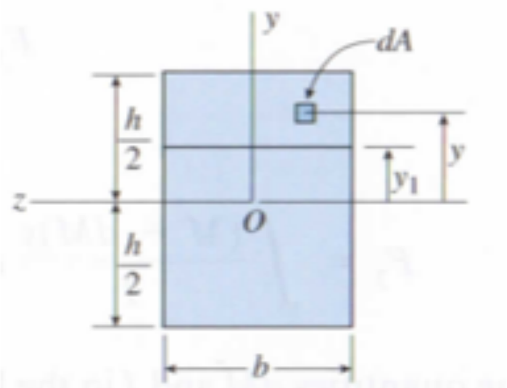
Side view of element



Side view of subelement

- If the bending is uniform, then there is no difference in moment on the face mp and the face m_1p_1. Therefore the normal forces cancel each other out.
- In non-uniform bending, there is a non-zero *dM* between the two cut planes.
- This means that $\sigma_1 \neq \sigma_2$. Which results in a force difference on the two faces.

Shear stress in beams



Cross section of beam at subelement

- We calculate the forces acting on the faces mp and m_1p_1 by integrating over the cross-sectional area of the sub-element.
- Since we are interested in the shear force at a specific value for $y=y_1$, we calculate the integral over the area of $b*y_1$ to $b*h/2$
- Due to the need for equilibrium in the x-direction, the difference between the two forces at the two sides of the sub-element has to be compensated by an additional longitudinal force, which is the shear force between two horizontal slices through the beam.

Shear stress in beams

- From the in-plane shear force we can then calculate the shear stress and find the shear formula:

$$\boxed{\tau = \frac{V \cdot Q}{I \cdot b}}$$

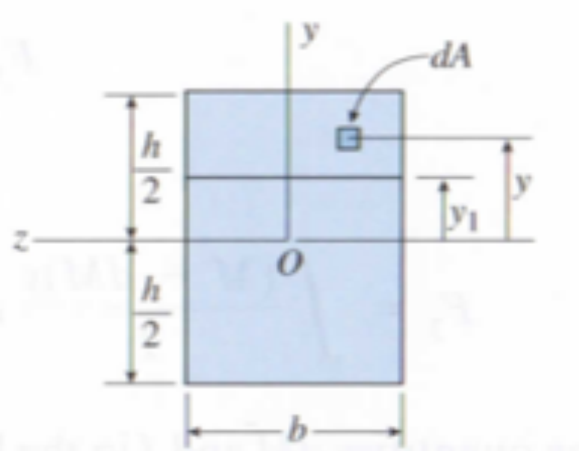
- with Q being the *first moment of area*:

$$\boxed{Q := \int_A y \cdot dA}$$

- The last thing we need to do to describe the shear stress distribution in the beam is to calculate Q.

Shear stress in beams

- For a beam with a rectangular cross-section, we can calculate $Q(y_1)$ by calculating the area from y_1 up to h/2
- We could also calculate the area from y_1 downward, and would bet –Q
- We can easily show that at $y_1=0$, Q is maximum and at $y_1=+/-\ h/2$, Q=0
- Through integration and combination with the shear formula we get:



Cross section of beam at subelement

$$\boxed{\tau(y) = \frac{V}{2I}\left(\frac{h^2}{4} - y^2\right)}$$

Shear stress in beams

$$\boxed{\tau(y) = \frac{V}{2I}\left(\frac{h^2}{4} - y^2\right)}$$

We see now:

- shear stress varies quadratically with the distance from the neutral axis
- shear stress is zero at the beams upper and lower surfaces
- the shear stress is maximum at its neutral axis:

$$\tau_{max} = \frac{Vh^2}{8I} = \frac{3V}{2A}$$

- the maximum shear stress is 50% higher than the average shear stress

Validity of the shear formula

During the derivation of the shear (and flexure) formula we have made a number of assumptions to make the derivation easier.

It is important to use the formula only in cases where these assumptions are justified:

- Edges of the cross-section must be parallel to the y axis
- The shear stresses must be uniform across the width of the cross-section
- The beam must be prismatic (e.g. must have a constant cross-section). The formula is not correct for a tapered beam.

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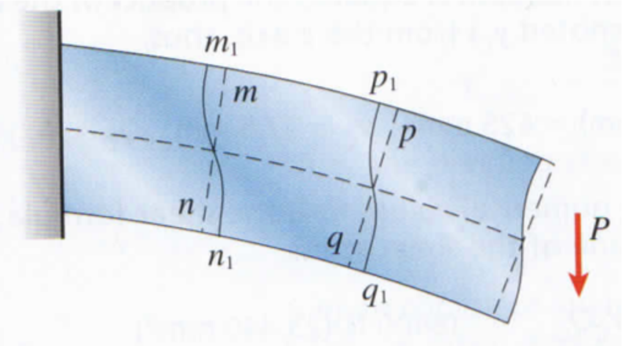
Effect of shear stress

- The shear stress distribution in the beam cross section is quadratic in the plane of bending.
- From Hooke's law in shear we know: $$\gamma = \frac{\tau}{G}$$
- Therefore also the shear strains are not constant along the cross-section. This means that plane cross-sections that were initially normal to the beam axis, will no longer be plane after bending!



Calculating shear and moment diagrams

Draw	Draw free body diagram
Calculate	calculate reaction forces
Make	Make a section through the beam and calculate V(x) and M(x) for that section
Draw	Draw the shear and moment diagram below the FBD
Check	Check if your diagram matches with what you know about the support